Cercul Judeţean de Excelenţă la Matematică **Prof. Maria Burlăciuc**

Constanţa **Şcoala Gimnazială "Tudor Arghezi"**

Năvodari

Lecţia 4 - clasa a VI-a

Evaluare

Evaluarea a fost făcută de prof. Maria Burlăciuc

Nr. Crt.	Numele şi prenumele elevului	A.8	A.10	A.22	A.23	A.26	A.27	A.30	Punctaj
1.	Albu Andrada	7	7	7	7	2	7	6	**43**
2.	Alexandru Bianca	7	7	7	7	7	7	6	**48**
3.	Baroană Ioan Costin	0	0	0	0	0	3	6	**9**
4.	Catană Alexandru	7	7	-	-	7	7	5	**33**
5.	Culică Tania	-	-	-	-	-	-	-	**-**
6.	Georgescu Tania	7	-	-	6	6	7	6	**32**
7.	Hiropedi Andreas	7	7	7	7	7	7	7	**49**
8.	Hristu Stelian	7	7	7	7	7	7	5	**47**
9.	Ibadula Ella Nelin	7	4	7	7	7	7	7	**46**
10.	Ion Cristian	-	-	-	-	-	-	-	**-**
11.	Jitea Octavian	5	7	-	7	6	7	7	**39**
12.	Manea Mircea	-	-	-	-	-	-	-	**-**
13.	Marin Mircea Mihai	6	3	2	7	7	7	2	**34**
14.	Mărăşescu Alexandru	-	-	-	-	-	-	7	**7**
15.	Memiş Edis	-	-	-	-	-	-	-	**-**
16.	Minea Alexandra	7	7	1	7	5	7	7	**41**
17.	Pariza Teodora	7	-	7	-	4	-	5	**23**
18.	Păşcălău Robert	7	7	7	7	7	7	7	**49**
19.	Pârvu Ioana Andreea	-	-	-	-	-	-	-	**-**
20.	Poşerba Dragoş	7	7	5	7	7	7	7	**47**
21.	Răduţ Florentina	-	-	-	-	-	-	-	**-**
22.	Resmeriţă Cristina	7	2	1	-	5	7	7	**29**
23.	Stan Alexandra	-	-	-	-	-	-	-	**-**
24.	Stoica Alexandru	3	-	-	7	6	7	7	**30**
25.	Tonca Tudor - Ştefan	-	-	-	-	-	-	-	**-**

Temă la algebră: ***Problemele 8,10, 22,23, 26, 27, 30***

8. ***Fie n∈N* şi*** $S_n = \frac{2}{3} - \frac{1}{2^2} + \frac{2}{3^2} - \frac{1}{2^3} + \frac{2}{3^3} - \frac{1}{2^4} + \ldots + \frac{2}{3^{n-1}} - \frac{1}{2^n} + \frac{2}{3^n}$. ***Să se demonstreze că*** $\frac{1}{2} < S_n < 1$.

Soluţie:

$$S_n = \frac{1}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{2^2} + \frac{2}{3^2} - \frac{1}{2^3} + \frac{2}{3^3} - \frac{1}{2^4} + \ldots + \frac{2}{3^{n-1}} - \frac{1}{2^n} + \frac{2}{3^n}$$

$$S_n = \frac{1}{2} + \left(\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots + \frac{2}{3^{n-1}} + \frac{2}{3^n}\right) - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots + \frac{1}{2^n}\right)$$

$$S_n = \frac{1}{2} + 2\cdot\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^{n-1}} + \frac{1}{3^n}\right) - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots + \frac{1}{2^n}\right)$$

Notez $a = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} \,|\cdot \frac{1}{3}$

$\frac{1}{3} \cdot a = \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$

$$a - \frac{a}{3} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} - \frac{1}{3^2} - \frac{1}{3^3} - \ldots - \frac{1}{3^n} - \frac{1}{3^{n+1}}$$

$$\frac{2a}{3} = \frac{1}{3} - \frac{1}{3^{n+1}} \,|\cdot 3$$

$2a = 1 - \frac{1}{3^n}$

Notez $b = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots + \frac{1}{2^n}\right) |\cdot \frac{1}{2}$

$\frac{b}{2} = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$

$b - \frac{b}{2} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots + \frac{1}{2^n} - \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} - \ldots - \frac{1}{2^n} - \frac{1}{2^{n+1}}$

$\frac{b}{2} = \frac{1}{2} - \frac{1}{2^{n+1}} \,|\cdot 2$

$b = 1 - \frac{1}{2^n}$

$$S_n = \frac{1}{2} + 1 - \frac{1}{3^n} - \left(1 - \frac{1}{2^n}\right) \Leftrightarrow S_n = \frac{1}{2} + 1 - \frac{1}{3^n} - 1 + \frac{1}{2^n} \Leftrightarrow S_n = \frac{1}{2} + \frac{1}{2^n} - \frac{1}{3^n}$$

$S_n = \frac{1}{2} + \frac{3^n - 2^n}{2^n \cdot 3^n} > \frac{1}{2}$ **(1)**

$n \in N^* \Rightarrow \frac{1}{2^n} \le \frac{1}{2} \Rightarrow \frac{1}{2^n} < \frac{1}{2} + \frac{1}{3^n} \Rightarrow \frac{1}{2^n} - \frac{1}{3^n} < \frac{1}{2} \Rightarrow \frac{1}{2} + \frac{1}{2^n} - \frac{1}{3^n} < \frac{1}{2} + \frac{1}{2} \Rightarrow S_n < 1$ **(2)**

Din relaţiile (1) şi (2) $\Rightarrow \frac{1}{2} < S_n < 1$

10. Să se demonstreze că pentru orice n ∈N, n≥ 3, avem:

$$\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots + \frac{1}{n^3} < \frac{1}{4}$$

Soluţie:

$n \in N^* \Rightarrow n^3 > n^3 - n \Leftrightarrow n^3 > n(n^2 - 1) \Leftrightarrow n^3 > n(n-1)(n+1) \Leftrightarrow n^3 > (n-1)\cdot n \cdot (n+1)$

$2^3 > 1 \cdot 2 \cdot 3 \Rightarrow \frac{1}{2^3} < \frac{1}{1 \cdot 2 \cdot 3}$

$3^3 > 2 \cdot 3 \cdot 4 \Rightarrow \frac{1}{3^3} < \frac{1}{2 \cdot 3 \cdot 4}$

$4^3 > 3 \cdot 4 \cdot 5 \Rightarrow \frac{1}{4^3} < \frac{1}{3 \cdot 4 \cdot 5}$

....

$n^3 > (n-1)\cdot n \cdot (n+1) \Rightarrow \frac{1}{n^3} < \frac{1}{(n-1)\cdot n \cdot (n+1)}$

Adunând relaţiile membru cu membru obţinem:

$$\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots+\frac{1}{n^3}<\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots+\frac{1}{(n-1)\cdot n\cdot(n+1)}$$

$$\frac{1}{2}\cdot\left[\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right]=\frac{1}{2}\cdot\frac{n+2-n}{n(n+1)(n+2)}=\frac{1}{2}\cdot\frac{2}{n(n+1)(n+2)}=\frac{1}{n(n+1)(n+2)}$$

$$\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots+\frac{1}{n^3}<\frac{1}{2}\cdot\left[\frac{1}{1\cdot2}-\frac{1}{2\cdot3}+\frac{1}{2\cdot3}-\frac{1}{3\cdot4}+\frac{1}{3\cdot4}-\frac{1}{4\cdot5}\ldots+\frac{1}{(n-1)\cdot n}-\frac{1}{n\cdot(n+1)}\right]$$

$$\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots+\frac{1}{n^3}<\frac{1}{2}\cdot\left[\frac{1}{1\cdot2}-\frac{1}{n\cdot(n+1)}\right]$$

$$\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots+\frac{1}{n^3}<\frac{1}{4}-\frac{1}{2n\cdot(n+1)}$$

$$\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots+\frac{1}{n^3}<\frac{1}{4}$$

22. Arătaţi că** $44<\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2010}{2009}<2011.$* ***(G.M. nr. 2/2010)

Soluţie:

Fie $N=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2009}{2010}$

$\frac{1}{2}>\frac{1}{3};\ \frac{3}{4}>\frac{3}{5};\ \frac{5}{6}>\frac{5}{7};\ldots;\ \frac{2009}{2010}>\frac{2009}{2011}\Rightarrow\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2009}{2010}>\frac{1}{3}\cdot\frac{3}{5}\cdot\frac{5}{7}\cdot\ldots\cdot\frac{2009}{2011}\Rightarrow$

$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2009}{2010}>\frac{1}{2011}\Rightarrow N>\frac{1}{2011}\Rightarrow\frac{1}{N}<2011\Rightarrow\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2010}{2009}<2011\ (1)$

Demonstrăm că $\frac{n}{n+1}<\frac{n+1}{n+2}$

$$\left.\begin{array}{l}\frac{n}{n+1}=\frac{n+1-1}{n+1}=\frac{n+1}{n+1}-\frac{1}{n+1}=1-\frac{1}{n+1}\\ \frac{n+1}{n+2}=\frac{n+2-1}{n+2}=\frac{n+2}{n+2}-\frac{1}{n+2}=1-\frac{1}{n+2}\\ n+1<n+2\Rightarrow\frac{1}{n+1}>\frac{1}{n+2}\end{array}\right\}\Rightarrow 1-\frac{1}{n+1}<1-\frac{1}{n+2}\Rightarrow\frac{n}{n+1}<\frac{n+1}{n+2}$$

$\frac{1}{2}<\frac{2}{3};\ \frac{3}{4}<\frac{4}{5};\ \frac{5}{6}<\frac{6}{7};\ldots;\ \frac{2009}{2010}<\frac{2010}{2011}\Rightarrow\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2009}{2010}<\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2010}{2011}\Rightarrow N<\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2010}{2011}\Rightarrow N<\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2010}{2009}\cdot\frac{1}{2011}\Rightarrow N<\frac{1}{N}\cdot\frac{1}{2011}\Rightarrow N^2<\frac{1}{2011}\Rightarrow N^2<\frac{1}{44^2}\Rightarrow$

$N<\frac{1}{44}\Rightarrow\frac{1}{N}>44\Rightarrow\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2010}{2009}>44\ (2)$

Din relațiile (1)și (2) $\Rightarrow 44<\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2010}{2009}<2011$

23. Fie** $x_1,x_2,\ldots,x_n>0,n\geq2$, **astfel încât:** $\frac{1}{x_1+1}+\frac{2}{x_2+1}+\frac{3}{x_3+1}+\ldots+\frac{n}{x_n+1}=1$. **Calculaţi în funcţie de n suma:** $\frac{x_1}{x_1+1}+\frac{2x_2}{x_2+1}+\ldots+\frac{nx_n}{x_n+1}$.* ***(G.M. nr. 12/2008)

Soluţie:

$$\frac{x_1}{x_1+1}+\frac{2x_2}{x_2+1}+\ldots+\frac{nx_n}{x_n+1}=\frac{x_1+1-1}{x_1+1}+\frac{2x_2+2-2}{x_2+1}+\ldots+\frac{nx_n+n-n}{x_n+1}=$$

$$\frac{x_1+1}{x_1+1}-\frac{1}{x_1+1}+\frac{2x_2+2}{x_2+1}-\frac{2}{x_2+1}+\ldots+\frac{nx_n+n}{x_n+1}-\frac{n}{x_n+1}=\frac{x_1+1}{x_1+1}-\frac{1}{x_1+1}+\frac{2(x_2+1)}{x_2+1}-$$

$$\frac{2}{x_2+1}+\ldots+\frac{n(x_n+1)}{x_n+1}-\frac{n}{x_n+1}=1+2+3+\ldots+n-\left(\frac{1}{x_1+1}+\frac{2}{x_2+1}+\frac{3}{x_3+1}+\ldots+\frac{n}{x_n+1}\right)=$$

$$= \frac{n(n+1)}{2} - 1 = \frac{n^2+n-2}{2}$$

26. Fie x= $\overline{0,0(ab)}$ ***, y =*** $\overline{0,0(0ab)}$ ***, z =*** $\overline{0,0(00ab)}$ ***cu a şi b cifre nenule în sistemul zecimal. Să se determine*** $\overline{ab}$ ***astfel încât:*** $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ ***să fie număr natural.***

Soluţie:

$$x = \frac{\overline{ab}}{990}\ ; y = \frac{\overline{ab}}{9990}; z = \frac{\overline{ab}}{99990} \Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{990}{\overline{ab}}+\frac{9990}{\overline{ab}}+\frac{99990}{\overline{ab}} = \frac{110970}{\overline{ab}} = \frac{2\cdot 3^4\cdot 5\cdot 137}{\overline{ab}} \in N$$

$$\Rightarrow \left.\begin{matrix} \overline{ab} \in D_{2\cdot 3^4\cdot 5\cdot 137} \\ a\ \text{și}\ b\ cifre\ nenule \end{matrix}\right\} \Rightarrow \overline{ab} \in \{15; 18; 27; 45; 54; 81\}$$

27. a) Demonstraţi că $\frac{1}{n(n+k)} = \frac{1}{k}\cdot\left(\frac{1}{n} - \frac{1}{n+k}\right)$ ***unde n, k*** $\in$***N****

b) Determinaţi numărul natural n, ştiind că: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{n(n+3)} = \frac{2012}{6039}$**.**

Soluţie:

a) $\frac{1}{k}\cdot\left(\frac{1}{n} - \frac{1}{n+k}\right) = \frac{1}{k}\cdot\frac{n+k-n}{n(n+k)} = \frac{1}{k}\cdot\frac{k}{n(n+k)} = \frac{1}{n(n+k)}$

b) Folosim $\frac{1}{n(n+k)} = \frac{1}{k}\cdot\left(\frac{1}{n} - \frac{1}{n+k}\right)$ ***unde n, k*** $\in$***N****

$$\frac{1}{3}\cdot\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\ldots+\frac{1}{n}-\frac{1}{n+3}\right) = \frac{2012}{6039}$$

$\frac{1}{3}\cdot\left(\frac{1}{1}-\frac{1}{n+3}\right) = \frac{2012}{6039} \mid \cdot 3 \Rightarrow \frac{1}{1}-\frac{1}{n+3} = \frac{2012}{2013} \Rightarrow \frac{1}{1} - \frac{2012}{2013} = \frac{1}{n+3} \Rightarrow \frac{1}{2013} = \frac{1}{n+3} \Rightarrow$ n+3=2013

$$n = 2010$$

30. Să se afle nnumerele naturale de trei cifre $\overline{xyz}$ ***cu proprietatea că*** $\frac{34}{x^2+y^2+z^2}$ ***este număr natural.***

Soluţie:

x, y, z ⇒ x, y, z ∈ {0; 1; 2; 3;. . .; 9} $și\ x \neq 0$

$$\frac{34}{x^2+y^2+z^2} \in N \Leftrightarrow (x^2+y^2+z^2)/34 \Leftrightarrow (x^2+y^2+z^2) \in D_{34} = \{1; 2; 17; 34\}$$

$$\left.\begin{matrix} x^2+y^2+z^2 = 1 \\ x \neq 0 \end{matrix}\right\} \Rightarrow (x; y; z) \in \{(1; 0; 0)\}$$

$$\left.\begin{matrix} x^2+y^2+z^2 = 2 \\ x \neq 0 \end{matrix}\right\} \Rightarrow (x; y; z) \in \{(1; 1; 0), (1; 0; 1)\}$$

$$\left.\begin{matrix} x^2+y^2+z^2 = 17 \\ x \neq 0 \end{matrix}\right\} \Rightarrow (x; y; z) \in \{(1; 4; 0), (1; 0; 4), (4; 1; 0), (4; 0; 1), (2; 2; 3), (2; 3; 2), (3; 2; 2)\}$$

$$\left.\begin{matrix} x^2+y^2+z^2 = 34 \\ x \neq 0 \end{matrix}\right\} \Rightarrow (x; y; z) \in \{(3; 3; 4), (3; 4; 3), (4; 3; 3), (3; 0; 5), (3; 5; 0), (5; 0; 3), (5; 3; 0)\}$$

$$\overline{xyz} \in \{100; 101; 104; 110; 140; 223; 232; 305; 322; 334; 343; 350; 401; 410; 433; 503; 530\}$$